Mamdani Control
Architecture of a Fuzzy Controller

- fuzzification interface
- decision logic
- defuzzification interface
- knowledge base
- measured values
- controller output
- not fuzzy
Example: Cartpole Problem (cont.)

$X_1$ is partitioned into 7 fuzzy sets.
Support of fuzzy sets: intervals with length $\frac{1}{4}$ of whole range $X_1$.
Similar fuzzy partitions for $X_2$ and $Y$.

**Next step:** specify rules

if $\xi_1$ is $A^{(1)}$ and $\ldots$ and $\xi_n$ is $A^{(n)}$ then $\eta$ is $B$,

$A^{(1)}, \ldots, A^{(n)}$ and $B$ represent linguistic terms corresponding to $\mu^{(1)}, \ldots, \mu^{(n)}$ and $\mu$ according to $X_1, \ldots, X_n$ and $Y$.

Let the rule base consist of $k$ rules.
Example: Cartpole Problem (cont.)

19 rules for cartpole problem, e.g.

If $\theta$ is approximately zero and $\dot{\theta}$ is negative medium then $F$ is positive medium.
Definition of Table-based Control Function

Measurement \((x_1, \ldots, x_n) \in X_1 \times \ldots \times X_n\) is forwarded to decision logic.

Consider rule

\[
\text{if } \xi_1 \text{ is } A^{(1)} \text{ and } \ldots \text{ and } \xi_n \text{ is } A^{(n)} \text{ then } \eta \text{ is } B.
\]

Decision logic computes degree to \(\xi_1, \ldots, \xi_n\) fulfills premise of rule.

For \(1 \leq \nu \leq n\), the value \(\mu^{(\nu)}(x_\nu)\) is calculated.

Combine values conjunctively by \(\alpha = \min \{\mu^{(1)}, \ldots, \mu^{(n)}\}\).

For each rule \(R_r\) with \(1 \leq r \leq k\), compute

\[
\alpha_r = \min \left\{ \mu_{i_1,r}^{(1)}(x_1), \ldots, \mu_{i_n,r}^{(n)}(x_n) \right\}.
\]
Output of $R_r = \text{fuzzy set of output values}$.  
Thus “cutting off” fuzzy set $\mu_{i_r}$ associated with conclusion of $R_r$ at $\alpha_r$.  
So for input $(x_1, \ldots, x_n)$, $R_r$ implies fuzzy set

$$\mu_{\text{output}}(R_r) : Y \rightarrow [0, 1],$$

$$y \mapsto \min \left\{ \mu_{i_1}^{(1)}(x_1), \ldots, \mu_{i_n}^{(n)}(x_n), \mu_{i_r}(y) \right\}.$$  

If $\mu_{i_1}^{(1)}(x_1) = \ldots = \mu_{i_n}^{(n)}(x_n) = 1$, then $\mu_{x_1, \ldots, x_n} = \mu_{i_r}$.  
If for all $\nu \in \{1, \ldots, n\}$, $\mu_{i_1}^{(\nu)}(x_\nu) = 0$, then $\mu_{x_1, \ldots, x_n} = 0$.  

R. Kruse, A. Dockhorn  
FS – Fuzzy Control  
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Combination of Rules

The decision logic combines the fuzzy sets from all rules. 

The maximum leads to the output fuzzy set

\[ \mu_{x_1, \ldots, x_n}^{\text{output}} : Y \rightarrow [0, 1], \]

\[ y \mapsto \max_{1 \leq r \leq k} \left\{ \min \left\{ \mu^{(1)}_{i_1, r}(x_1), \ldots, \mu^{(n)}_{i_n, r}(x_n), \mu_{ir}(y) \right\} \right\}. \]

Then \( \mu_{x_1, \ldots, x_n}^{\text{output}} \) is passed to defuzzification interface.
Rule evaluation for Mamdani-Assilian controller.

Input tuple \((25, -4)\) leads to fuzzy output.

Crisp output is determined by defuzzification.
Defuzzification

So far: mapping between each \((n_1, \ldots, n_n)\) and \(\mu_{x_1, \ldots, x_n}^{\text{output}}\).

Output = description of output value as fuzzy set.

Defuzzification interface derives crisp value from \(\mu_{x_1, \ldots, x_n}^{\text{output}}\).

This step is called **defuzzification**.

Most common methods:

- max criterion,
- mean of maxima,
- center of gravity.
The Max Criterion Method

Choose an arbitrary $y \in Y$ for which $\mu_{x_1,\ldots,x_n}^{\text{output}}$ reaches the maximum membership value.

Advantages:

- Applicable for arbitrary fuzzy sets.
- Applicable for arbitrary domain $Y$ (even for $Y \neq \mathbb{R}$).

Disadvantages:

- Rather class of defuzzification strategies than single method.
- Which value of maximum membership?
- Random values and thus non-deterministic controller.
- Leads to discontinuous control actions.
The Mean of Maxima (MOM) Method

Preconditions:

(i) \( Y \) is interval

(ii) \( Y_{\text{Max}} = \{ y \in Y \mid \forall y' \in Y : \mu_{x_1,\ldots,x_n}^{\text{output}}(y') \leq \mu_{x_1,\ldots,x_n}^{\text{output}}(y) \} \) is non-empty and measurable

(iii) \( Y_{\text{Max}} \) is set of all \( y \in Y \) such that \( \mu_{x_1,\ldots,x_n}^{\text{output}} \) is maximal

Crisp output value = mean value of \( Y_{\text{Max}} \).

- if \( Y_{\text{Max}} \) is finite:
  \[
  \eta = \frac{1}{|Y_{\text{Max}}|} \sum_{y_i \in Y_{\text{Max}}} y_i
  \]

- if \( Y_{\text{Max}} \) is infinite:
  \[
  \eta = \frac{\int_{y \in Y_{\text{Max}}} y \, dy}{\int_{y \in Y_{\text{Max}}} dy}
  \]

MOM can lead to discontinuous control actions.
Center of Gravity (COG) Method

Same preconditions as MOM method.

\[ \eta = \text{center of gravity/area of } \mu_{x_1, \ldots, x_n} \]

If \( Y \) is finite, then

\[ \eta = \frac{\sum_{y_i \in Y} y_i \cdot \mu_{x_1, \ldots, x_n}(y_i)}{\sum_{y_i \in Y} \mu_{x_1, \ldots, x_n}(y_i)}. \]

If \( Y \) is infinite, then

\[ \eta = \frac{\int_{y \in Y} y \cdot \mu_{x_1, \ldots, x_n}(y) \, dy}{\int_{y \in Y} \mu_{x_1, \ldots, x_n}(y) \, dy}. \]
Center of Gravity (COG) Method

Advantages:

- Nearly always smooth behavior,
- If certain rule dominates once, not necessarily dominating again.

Disadvantage:

- No semantic justification,
- Long computation,
- Counterintuitive results possible.

Also called center of area (COA) method:

take value that splits $\mu_{x_1,...,x_n}^{\text{output}}$ into 2 equal parts.
Example

Task: compute $\eta_{\text{COG}}$ and $\eta_{\text{MOM}}$ of fuzzy set shown below.

Based on finite set $Y = 0, 1, \ldots, 10$ and infinite set $Y = [0, 10]$. 

\text{Graph with \(\mu_{x_1, \ldots, x_n}(y)\), \(\eta_{\text{COG}}\), and \(\eta_{\text{MOM}}\).}
Example for COG
Continuous and Discrete Output Space

\[ \eta_{\text{COG}} = \frac{\int_{0}^{10} y \cdot \mu_{x_1,\ldots,x_n}^{\text{output}}(y) \, dy}{\int_{0}^{10} \mu_{x_1,\ldots,x_n}^{\text{output}}(y) \, dy} \]

\[ = \frac{\int_{0}^{5} 0.4y \, dy + \int_{5}^{7} (0.2y - 0.6)y \, dy + \int_{7}^{10} 0.8y \, dy}{5 \cdot 0.4 + 2 \cdot \frac{0.8 + 0.4}{2} + 3 \cdot 0.8} \]

\[ \approx \frac{38.7333}{5.6} \approx 6.917 \]

\[ \eta_{\text{COG}} = \frac{0.4 \cdot (0 + 1 + 2 + 3 + 4 + 5) + 0.6 \cdot 6 + 0.8 \cdot (7 + 8 + 9 + 10)}{0.4 \cdot 6 + 0.6 \cdot 1 + 0.8 \cdot 4} \]

\[ = \frac{36.8}{6.2} \approx 5.935 \]
Example for MOM
Continuous and Discrete Output Space

\[ \eta_{MOM} = \frac{\int_7^{10} y \, dy}{\int_7^{10} dy} \]

\[ = \frac{50 - 24.5}{10 - 7} = \frac{25.5}{3} \]

\[ = 8.5 \]

\[ \eta_{MOM} = \frac{7 + 8 + 9 + 10}{4} \]

\[ = \frac{34}{4} \]

\[ = 8.5 \]
Problem Case for MOM and COG

What would be the output of MOM or COG?
Is this desirable or not?
Example: Engine Idle Speed Control
VW 2000cc 116hp Motor (Golf GTI)

from the air cleaner
air flow sensor
banking up flap

air bypass to the throttle

auxiliary air regulator
DIGIFANT control device

compensation flap to damp vibrations
to the intake valves
Structure of the Fuzzy Controller

fuzzy controller

AARSREV
REV0_LO
dAIRCON

data prep.

state detect. and MFC activ.

meta controller
dREV
gREV
dAARCUR

control range limit.

pilot value for air conditioning system

AARCURIN
Deviation of the Number of Revolutions \(d_{REV}\)
Gradient of the Number of Revolutions

\( g_{REV} \)
Change of Current for Auxiliary Air Regulator
dAARCUR
Rule Base

If the deviation from the desired number of revolutions is negative small and the gradient is negative medium, then the change of the current for the auxiliary air regulation should be positive medium.

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<th>az</th>
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Performance Characteristics

dAARCUR

-70 70 40 -40

dREV 70 40 gREV

-20 -15 -10 -5 0 5 10 15 20
Example: Automatic Gear Box I

VW gear box with 2 modes (eco, sport) in series line until 1994.
Research issue since 1991: individual adaption of set points and no additional sensors.

Idea: car “watches” driver and classifies him/her into calm, normal, sportive (assign sport factor [0, 1]), or nervous (calm down driver).

Test car: different drivers, classification by expert (passenger).
Simultaneous measurement of 14 attributes, e.g., speed, position of accelerator pedal, speed of accelerator pedal, kick down, steering wheel angle.
Example: Automatic Gear Box II
Continuously Adapting Gear Shift Schedule in VW New Beetle

**Classification of driver / driving situation by fuzzy logic**

- Fuzzification
  - Accelerator pedal
  - Filtered speed of accelerator pedal
  - Number of changes in pedal direction
  - Sport factor [t-1]

- Inference machine
- Rule base

**Defuzzification**
- Sport factor [t]

**Gear shift computation**

- Interpolation
  - Determination of speed limits for shifting into higher or lower gear depending on sport factor

**Gear selection**
Example: Automatic Gear Box III
Technical Details

Optimized program on Digimat:
24 byte RAM
702 byte ROM
uses 7 Mamdani fuzzy rules

Runtime: 80 ms
12 times per second new sport factor is assigned.

Research topics:
When fuzzy control?
How to find fuzzy rules?
Takagi Sugeno Control
Takagi-Sugeno Controller

Proposed by Tomohiro Takagi and Michio Sugeno.

Modification/extension of Mamdani controller.

Both in common: fuzzy partitions of input domain \( X_1, \ldots, X_n \).

Difference to Mamdani controller:
- no fuzzy partition of output domain \( Y \),
- controller rules \( R_1, \ldots, R_k \) are given by

\[
R_r : \text{if} \ \xi_1 \text{ is } A^{(1)}_{i_1, r} \text{ and } \ldots \text{ and } \xi_n \text{ is } A^{(n)}_{i_n, r} \text{ then } \eta_r = f_r(\xi_1, \ldots, \xi_n),
\]

\[
f_r : X_1 \times \ldots \times X_n \rightarrow Y.
\]
- Generally, \( f_r \) is linear, i.e. \( f_r(x_1, \ldots, x_n) = a_0^{(r)} + \sum_{i=1}^{n} a_i^{(r)} x_i \).
For given input \((x_1, \ldots, x_n)\) and for each \(R_r\), decision logic computes truth value \(\alpha_r\) of each premise, and then \(f_r(x_1, \ldots, x_n)\).

Analogously to Mamdani controller:

\[
\alpha_r = \min \left\{ \mu_{i_1,r}(x_1), \ldots, \mu_{i_n,r}(x_n) \right\}.
\]

Output equals crisp control value

\[
\eta = \frac{\sum_{r=1}^{k} \alpha_r \cdot f_r(x_1, \ldots, x_n)}{\sum_{r=1}^{k} \alpha_r}.
\]

Thus no defuzzification method necessary.
Example

\[ R_1 : \text{if } \xi_1 \text{ is } 3 \rightarrow 9 \text{ then } \eta_1 = 1 \cdot \xi_1 + 0.5 \cdot \xi_2 + 1 \]

\[ R_2 : \text{if } \xi_1 \text{ is } 3 \rightarrow 9 \text{ and } \xi_2 \text{ is } 4 \rightarrow 13 \text{ then } \eta_2 = -0.1 \cdot \xi_1 + 4 \cdot \xi_2 + 1.2 \]

\[ R_3 : \text{if } \xi_1 \text{ is } 3 \rightarrow 11 \rightarrow 18 \text{ and } \xi_2 \text{ is } 4 \rightarrow 13 \text{ then } \eta_3 = 0.9 \cdot \xi_1 + 0.7 \cdot \xi_2 + 9 \]

\[ R_4 : \text{if } \xi_1 \text{ is } 11 \rightarrow 18 \text{ and } \xi_2 \text{ is } 4 \rightarrow 13 \text{ then } \eta_4 = 0.2 \cdot \xi_1 + 0.1 \cdot \xi_2 + 0.2 \]

If a certain clause “\(x_j \text{ is } A_{ij,r}^{(j)}\)” in rule \(R_r\) is missing, then \(\mu_{ij,r}(x_j) \equiv 1\) for all linguistic values \(i_{j,r}\).

For instance, here \(x_2\) in \(R_1\), so \(\mu_{i_{2,1}}(x_2) \equiv 1\) for all \(i_{2,1}\).
Example: Output Computation

input: \((\xi_1, \xi_2) = (6, 7)\)

\[
\begin{align*}
\alpha_1 &= \frac{1}{2} \land 1 = \frac{1}{2} & \eta_1 &= 6 + \frac{7}{2} + 1 = 10.5 \\
\alpha_2 &= \frac{1}{2} \land \frac{2}{3} = \frac{1}{2} & \eta_2 &= -0.6 + 28 + 1.2 = 28.6 \\
\alpha_3 &= \frac{1}{2} \land \frac{1}{3} = \frac{1}{3} & \eta_3 &= 0.9 \cdot 6 + 0.7 \cdot 7 + 9 = 19.3 \\
\alpha_4 &= 0 \land \frac{1}{3} = 0 & \eta_4 &= 6 + \frac{7}{2} + 1 = 10.5
\end{align*}
\]

output: \(\eta = f(6, 7) = \frac{\frac{1}{2} \cdot 10.5 + \frac{1}{2} \cdot 28.6 + \frac{1}{3} \cdot 19.3}{\frac{1}{2} + \frac{1}{2} + \frac{1}{3}} = 19.5\)
Example: Passing a Bend

Pass a bend with a car at constant speed.

Measured inputs:
\( \xi_1 \): distance of car to beginning of bend
\( \xi_2 \): distance of car to inner barrier
\( \xi_3 \): direction (angle) of car
\( \xi_4 \): distance of car to outer barrier

\( \eta \) = rotation speed of steering wheel

\( X_1 = [0 \text{ cm}, 150 \text{ cm}] \), \( X_2 = [0 \text{ cm}, 150 \text{ cm}] \)

\( X_3 = [-90^\circ, 90^\circ] \), \( X_4 = [0 \text{ cm}, 150 \text{ cm}] \)
Fuzzy Partitions of $X_1$ and $X_2$
Fuzzy Partitions of $X_3$ and $X_4$
Form of Rules of Car

$$R_r : \text{if } \xi_1 \text{ is } A \text{ and } \xi_2 \text{ is } B \text{ and } \xi_3 \text{ is } C \text{ and } \xi_4 \text{ is } D$$

then $$\eta = p_0^{(A,B,C,D)} + p_1^{(A,B,C,D)} \cdot \xi_1 + p_2^{(A,B,C,D)} \cdot \xi_2 + p_3^{(A,B,C,D)} \cdot \xi_3 + p_4^{(A,B,C,D)} \cdot \xi_4$$

$$A \in \{\text{small, medium, big}\}$$

$$B \in \{\text{small, big}\}$$

$$C \in \{\text{outwards, forward, inwards}\}$$

$$D \in \{\text{small}\}$$

$$p_0^{(A,B,C,D)}, \ldots, p_4^{(A,B,C,D)} \in \mathbb{R}$$
Control Rules for the Car

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Sample Calculation

Assume that the car is 10 cm away from beginning of bend ($\xi_1 = 10$).
The distance of the car to the inner barrier be 30 cm ($\xi_2 = 30$).
The distance of the car to the outer barrier be 50 cm ($\xi_4 = 50$).
The direction of the car be “forward” ($\xi_3 = 0$).

Then according to all rules $R_1, \ldots, R_{20}$, only premises of $R_4$ and $R_7$ have a value $\neq 0$. 
## Membership Degrees to Control Car

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<tr>
<th></th>
<th>small</th>
<th>medium</th>
<th>big</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\xi_1 = 10$</td>
<td>0.8</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>small</th>
<th>big</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\xi_2 = 30$</td>
<td>0.25</td>
<td>0.167</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>outwards</th>
<th>forward</th>
<th>inwards</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\xi_3 = 0$</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>small</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\xi_4 = 50$</td>
<td>0</td>
</tr>
</tbody>
</table>
Sample Calculation (cont.)

For the premise of $R_4$ and $R_7$, $\alpha_4 = \frac{1}{4}$ and $\alpha_7 = \frac{1}{6}$, resp.

The rules weights $\alpha_4 = \frac{\frac{1}{4}}{\frac{1}{4} + \frac{1}{6}} = \frac{3}{5}$ for $R_4$ and $\alpha_5 = \frac{2}{5}$ for $R_7$.

$R_4$ yields

$$\eta_4 = 0.303 - 0.026 \cdot 10 + 0.061 \cdot 30 - 0.050 \cdot 0 + 0.000 \cdot 50$$
$$= 1.873.$$

$R_7$ yields

$$\eta_7 = 2.990 - 0.017 \cdot 10 + 0.000 \cdot 30 - 0.021 \cdot 0 + 0.000 \cdot 50$$
$$= 2.820.$$

The final value for control variable is thus

$$\eta = \frac{3}{5} \cdot 1.873 + \frac{2}{5} \cdot 2.820 = 2.2518.$$
Fuzzy Control as Similarity-Based reasoning
Interpolation in the Presence of Fuzziness

Both Takagi-Sugeno and Mamdani are based on heuristics. They are used without a concrete interpretation.

Fuzzy control is interpreted as a method to specify a non-linear transition function by knowledge-based interpolation.

A fuzzy controller can be interpreted as fuzzy interpolation.

Now recall the concept of fuzzy equivalence relations (also called similarity relations).
Similarity: An Example

Specification of a partial control mapping (“good control actions”):

<table>
<thead>
<tr>
<th>deviation</th>
<th>gradient</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>-40.0</td>
</tr>
<tr>
<td>-70.0</td>
<td>22.5</td>
</tr>
<tr>
<td>-50.0</td>
<td>22.5</td>
</tr>
<tr>
<td>-30.0</td>
<td>15.0</td>
</tr>
<tr>
<td>0.0</td>
<td>5.0</td>
</tr>
<tr>
<td>30.0</td>
<td>0.0</td>
</tr>
<tr>
<td>50.0</td>
<td>0.0</td>
</tr>
<tr>
<td>70.0</td>
<td>-5.0</td>
</tr>
</tbody>
</table>
Interpolation of Control Table

There might be additional knowledge available:

Some values are “indistinguishable”, “similar” or “approximately equal”.

Or they should be treated in a similar way.

Two problems:

a) How to model information about similarity?
b) How to interpolate in case of an existing similarity information?
How to Model Similarity?
Proposal 1: Equivalence Relation

Definition

Let $A$ be a set and $\approx$ be a binary relation on $A$. $\approx$ is called an equivalence relation if and only if $\forall a, b, c \in A$,

(i) $a \approx a$ (reflexivity)
(ii) $a \approx b \iff b \approx a$ (symmetry)
(iii) $a \approx b \land b \approx c \rightarrow a \approx c$ (transitivity).

Let us try $a \approx b \iff |a - b| < \varepsilon$ where $\varepsilon$ is fixed.

$\approx$ is not transitive, $\approx$ is no equivalence relation.

Recall the Poincaré paradox: $a \approx b, b \approx c, a \not\approx c$.

This is counterintuitive.
How to Model Similarity?
Proposal 2: Fuzzy Equivalence Relation

Definition

A function $E : X^2 \to [0, 1]$ is called a fuzzy equivalence relation with respect to the t-norm $\top$ if it satisfies the following conditions

(i) $E(x, x) = 1$  \hspace{1cm} \text{(reflexivity)}

(ii) $E(x, y) = E(y, x)$  \hspace{1cm} \text{(symmetry)}

(iii) $\top(E(x, y), E(y, z)) \leq E(x, z)$  \hspace{1cm} \text{($t$-transitivity)}.

$E(x, y)$ is the degree to which $x \approx y$ holds.

$E$ is also called similarity relation, $t$-equivalence relation, indistinguishability operator, or tolerance relation.

Note that property (iii) corresponds to the vague statement if $(x \approx y) \land (y \approx z)$ then $x \approx z$. 
Fuzzy Equivalence Relations: An Example

Let $\delta$ be a pseudo metric on $X$.

Furthermore $\top(a, b) = \max\{a + b - 1, 0\}$ Łukasiewicz $t$-norm.

Then $E_\delta(x, y) = 1 - \min\{\delta(x, y), 1\}$ is a fuzzy equivalence relation.

$\delta(x, y) = 1 - E_\delta(x, y)$ is the induced pseudo metric.

Here, fuzzy equivalence and distance are dual notions in this case.

**Definition**

A function $E : X^2 \rightarrow [0, 1]$ is called a fuzzy equivalence relation if

For all $x, y, z \in X$

(i) $E(x, x) = 1$ (reflexivity)

(ii) $E(x, y) = E(y, x)$ (symmetry)

(iii) $\max\{E(x, y) + E(y, z) - 1, 0\} \leq E(x, z)$ (Łukasiewicz transitivity).
Fuzzy Sets as Derived Concept

\[ \delta(x, y) = |x - y| \quad \text{metric} \]
\[ E_\delta(x, y) = 1 - \min\{|x - y|, 1\} \quad \text{fuzzy equivalence relation} \]

\[ \mu_{x_0} : X \rightarrow [0, 1] \]
\[ x \mapsto E_\delta(x, x_0) \quad \text{fuzzy singleton} \]

\( \mu_{x_0} \) describes “local” similarities.
Extensional Hull

\[ E : \mathbb{IR} \times \mathbb{IR} \to [0, 1], \quad (x, y) \mapsto 1 - \min\{|x - y|, 1\} \text{ is fuzzy equivalence relation w.r.t. } \top_{\text{Łuka}}. \]

**Definition**

Let \( E \) be a fuzzy equivalence relation on \( X \) w.r.t. \( \top \).

\( \mu \in \mathcal{F}(X) \) is extensional if and only if

\[ \forall x, y \in X : \top(\mu(x), E(x, y)) \leq \mu(y). \]

**Definition**

Let \( E \) be a fuzzy equivalence relation on a set \( X \).

Then the extensional hull of a set \( M \subseteq X \) is the fuzzy set

\[ \mu_M : X \to [0, 1], \quad x \mapsto \sup\{E(x, y) \mid y \in M\}. \]

The extensional hull of \( \{x_0\} \) is called a singleton.
Specification of Fuzzy Equivalence Relation

Given a family of fuzzy sets that describes “local” similarities.

There exists a fuzzy equivalence relation on \( X \) with induced singletons \( \mu_i \) if and only if

\[
\forall i, j : \sup_{x \in X} \{\mu_i(x) + \mu_j(x) - 1\} \leq \inf_{y \in X} \{1 - |\mu_i(y) - \mu_j(y)|\}.
\]

If \( \mu_i(x) + \mu_j(x) \leq 1 \) for \( i \neq j \), then there is a fuzzy equivalence relation \( E \) on \( X \)

\[
E(x, y) = \inf_{i \in I} \{1 - |\mu_i(x) - \mu_i(y)|\}.
\]
Necessity of Scaling I

Are there other fuzzy equivalence relations on \( \mathbb{R} \) than
\[ E(x, y) = 1 - \min\{|x - y|, 1\}? \]

Integration of scaling.

A fuzzy equivalence relation depends on the measurement unit, e.g.
- Celsius: \( E(20 ^\circ C, 20.5 ^\circ C) = 0.5 \),
- Fahrenheit: \( E(68 F, 68.9 F) = 0.9 \),
- scaling factor for Celsius/Fahrenheit = 1.8 \((F = 9/5C + 32)\).

\[ E(x, y) = 1 - \min\{|c \cdot x - c \cdot y|, 1\} \] is a fuzzy equivalence relation!
Necessity of Scaling II

How to generalize scaling concept?

\[ X = [a, b]. \]

Scaling \( c : X \to [0, \infty) \).

Transformation

\[
 f : X \to [0, \infty), \quad x \mapsto \int_a^x c(t)dt.
\]

Fuzzy equivalence relation

\[
 E : X \times X \to [0, 1], \quad (x, y) \mapsto 1 - \min\{|f(x) - f(y)|, 1\}.
\]
The imprecision of measurements is modeled by a fuzzy equivalence relations $E_1, \ldots, E_n$ and $F$ on $X_1, \ldots, X_n$ and $Y$, resp.

The information provided by control expert are

- $k$ input-output tuples $(x_1^{(r)}, \ldots, x_n^{(r)}, y^{(r)})$ and
- the description of the fuzzy equivalence relations for input and output spaces, resp.

The goal is to derive a control function $\varphi : X_1 \times \ldots \times X_n \rightarrow Y$ from this information.
Determine Fuzzy-valued Control Functions I

The extensional hull of graph of $\varphi$ must be determined. Then the equivalence relation on $X_1 \times \ldots \times X_n \times Y$ is

$$E(((x_1, \ldots, x_n, y), (x_1', \ldots, x_n', y')) = \min\{E_1(x_1, x_1'), \ldots, E_n(x_n, x_n'), F(y, y')\}.$$
Determine Fuzzy-valued Control Functions II

For $X_i$ and $Y$, define the sets

\[
X_i^{(0)} = \left\{ x \in X_i \mid \exists r \in \{1, \ldots, k\} : x = x_i^{(r)} \right\}
\]

and

\[
Y^{(0)} = \left\{ y \in Y \mid \exists r \in \{1, \ldots, k\} : y = y^{(r)} \right\}.
\]

$X_i^{(0)}$ and $Y^{(0)}$ contain all values of the $r$ input-output tuples $(x_1^{(r)}, \ldots, x_n^{(r)}, y^{(r)})$.

For each $x_0 \in X_i^{(0)}$, singleton $\mu_{x_0}$ is obtained by

\[
\mu_{x_0}(x) = E_i(x, x_0).
\]
If $\varphi$ is only partly given, then use $E_1, \ldots, E_n, F$ to fill the gaps of $\varphi_0$. The extensional hull of $\varphi_0$ is a fuzzy set

$$
\mu_{\varphi_0}(x'_1, \ldots, x'_n, y') = \max_{r \in \{1, \ldots, k\}} \left\{ \min \left\{ E_1(x_1^{(r)}, x'_1), \ldots, E_n(x_n^{(r)}, x'_n), F(y^{(r)}, y') \right\} \right\}.
$$

$\mu_{\varphi_0}$ is the smallest fuzzy set containing the graph of $\varphi_0$. Obviously, $\mu_{\varphi_0} \leq \mu_{\varphi}$

$$
\mu_{\varphi_0}^{(x_1, \ldots, x_n)} : Y \rightarrow [0, 1],
$$

$$
y \mapsto \mu_{\varphi_0}(x_1, \ldots, x_n, y).
$$
Reinterpretation of Mamdani Controller

For input \((x_1, \ldots, x_n)\), the projection of the extensional hull of graph of \(\varphi_0\) leads to a fuzzy set as output.

This is identical to the Mamdani controller output.

It identifies the input-output tuples of \(\varphi_0\) by linguistic rules:

\[
R_r : \text{if } x_1 \text{ is approximately } x_1^{(r)} \\
\text{and } \ldots \\
\text{and } x_n \text{ is approximately } x_n^{(r)} \\
\text{then } y \text{ is } y^{(r)}. 
\]

A fuzzy controller based on equivalence relations behaves like a Mamdani controller.
3 fuzzy rules (specified by 3 input-output tuples).

The extensional hull is the maximum of all fuzzy rules.
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